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Isotropic Coordinates and Schwarzschild Metric

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Written in terms of isotropic coordinates r, t, the Schwarzschild metric as usually given is static, i.e., admits a timelike Killing vector for all values of r and t. Therefore the region within the event horizon cannot be accounted for. This deficiency is remedied here, by finding the general spherically symmetric vacuum metric in isotropic coordinates.

1. INTRODUCTION

The Schwarzschild metric, i.e., the spherically symmetric vacuum solution of Einstein's equations, is traditionally derived (e.g., Rindler, 1977) in canonical, or curvature, coordinates (R, θ, ϕ, T):

$$ds^2 = -\gamma^{-1}\, dR^2 - R^2\, d\omega^2 + \gamma\, dT^2 \tag{1}$$

where $d\omega^2 := d\theta^2 + \sin^2\theta\, d\phi^2$, $\gamma := 1 - 2M/R$, and M is a constant of integration, taken as nonnegative for physical reasons. When the metric is to be exhibited in some other coordinate system this is usually, though not always (e.g., Buchdahl, 1981), achieved by appropriate transformation of (1). A frequently recurring example of this is the case of isotropic coordinates (r, θ, ϕ, t) which are such that the metric has the generic form

$$ds^2 = -A^2(dr^2 + r^2\, d\omega^2) + B^2\, dt^2 \tag{2a}$$

Almost without exception the required transformation is given without further comment (e.g., Adler, Bazin, and Schiffer, 1965; Anderson, 1967; Landau and Lifshitz, 1975; Misner, Thorne, and Wheeler, 1973; Papapetrou, 1974; Robertson and Noonan, 1968; Synge, 1969; Tolman, 1934; Weinberg, 1972) to be

$$R = r(1 + M/2r)^2, \qquad T = t \tag{3}$$

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which makes

$$A^2=(1+M/2r)^4, \qquad B^2=(1-M/2r)^2(1+M/2r)^{-2} \tag{2b}$$

However, (3) is evidently deficient in that it implies a least value ($=2M$) of R, when $r=\frac{1}{2}M$, concomitantly with the invariance of (3) and therefore of (2a) under the inversion $r=M^2/4r'$. As r goes from ∞ to 0 the region $\mathscr{E}$ outside the event horizon $\mathscr{H}$ is covered twice; and the region $\mathscr{I}$ within $\mathscr{H}$ is unaccounted for. Of course, this was to be expected since, whereas (1) does not admit a timelike Killing vector in $\mathscr{I}$, any metric (2a) with t-independent A and B is static and so cannot cover $\mathscr{I}$. In short, instead of seeking a transformation $R=R(r)$, $T=t$ of (1) one has to look for a more general transformation

$$R=R(r,t), \qquad T=T(r,t) \tag{4}$$

which takes (1) into (2a), with A and B now functions of r and t. This is achieved in Section 2. Whereas $R(r,t)$ is comparatively simple, $T(r,t)$ is rather less tractable. In Section 3 the functions A and B are obtained explicitly. Finally, Section 4 concerns the event horizon in some detail with particular reference to the existence of coordinate singularities.

2. THE FUNCTIONS $R(r,t)$ AND $T(r,t)$

The transformation (4) will take (1) into a metric of the generic form (2) if

$$\gamma^{-1}R_r^2-\gamma T_r^2=A^2 \tag{5}$$

$$-\gamma^{-1}R_t^2+\gamma T_t^2=B^2 \tag{6}$$

$$\gamma^{-1}R_rT_t-\gamma T_rT_t=0 \tag{7}$$

$$R=rA \tag{8}$$

where subscripts denote partial derivatives. Eliminating T_r and T_t between the first three of these, there comes

$$A^{-2}R_r^2-B^{-2}R_t^2=\gamma \tag{9}$$

At this point it is advantageous to fall back on the field equation $T_1{}^4=0$, i.e. (Tolman, 1934),

$$(\ln A)_{rt}-(\ln A)_t(\ln B)_r=0$$

which implies that

$$A_t/A=f(t)B \tag{10}$$

where $f(t)$ is an arbitrary (nonconstant) function of t whose range is required to be $0 \leq f \leq \infty$. An equation for R now follows on eliminating A and B from (9) by means of (8) and (10):

$$R_x^2 = -2MR + R^2 + f^2R^4 \tag{11}$$

where $x := \ln r$. (It follows from the equations above that A^2 and B^2 are nonnegative, whatever the sign of γ.)

Since R_x^2 and R are nonnegative, (11) requires that

$$f^2R^3 + R - 2M \geq 0 \tag{12}$$

If $\chi := Mf$ this implies that $R \geq q/f$, where q is the real root of the equation

$$q^3 + q = 2\chi \tag{13}$$

i.e.,

$$q = [(\chi^2 + \tfrac{1}{27})^{1/2} + \chi]^{1/3} - [(\chi^2 + \tfrac{1}{27})^{1/2} - \chi]^{1/3} \tag{14}$$

R is evidently an elliptic function, granted that $M \neq 0$. In fact, equation (11) is satisfied by

$$R(r, t) = Mk\frac{\mathbf{c}+1}{\mathbf{c}-\lambda} \tag{15}$$

where

$$\mathbf{c} := \mathrm{cn}(hx + b|m) \tag{16}$$

Here b is an as yet arbitrary function of t and λ, m, h, k depend on t alone: given a value of χ the value of q is obtained from (14) and that of χ in turn from the relation

$$q^2 = 2\lambda/(1 - 4\lambda + \lambda^2) \tag{17a}$$

Then

$$m = (1 - \tfrac{1}{2}\lambda)/(1 - \lambda^2) \tag{17b}$$

$$h = [(1 - \lambda^2)/(1 - 4\lambda + \lambda^2)]^{1/2} \tag{17c}$$

$$k = (1 - 4\lambda + \lambda^2)/(1 - \lambda) \tag{17d}$$

The behavior of these functions is briefly analyzed in Appendix A.

It is advantageous to write $b := h \ln a$ so that the argument $hx + b$ is $h \ln(ar) := \xi$, say. In fact $a(t)$ must be a constant, as will be shown at the end of this section. As a matter of convenience I shall take $a = 1$, except where otherwise indicated. Further, it is already clear from equation (11) that R is indifferent to the sign of x. Consequently it suffices to adopt the convention $r \geq 1$. (Note that invariance under sign reversal of x amounts to invariance of ds^2 under the inversion $r \to 1/r$.)

By inspection of (14) and (17a), λ only goes over the narrow range $0 \leq \lambda \leq 2-\sqrt{3}(:=\lambda^*)$ as χ goes from 0 to ∞. R takes its least value

$$R_0 := 2Mk/(1-\lambda) \tag{18}$$

when $\mathbf{c}=1$. Therefore

$$\gamma_0 := -12M/R_0 = -q^2 \tag{19}$$

This value of γ clearly belongs to points of $\mathcal{I}$. More generally points of $\mathcal{I}$ have $R<2M$, i.e.,

$$\mathbf{c} > (1-2\lambda-\lambda^2)/(1+2\lambda-\lambda^2) := \kappa \tag{20}$$

According to (15) R can take values arbitrarily close to zero only if k is sufficiently small. $k \to 0$, however, implies that $\lambda \to \lambda^*$ and $\chi \to \infty$. Therefore, retaining only dominant terms in (A2), (20) becomes, with $m^* = \frac{1}{4}(2+\sqrt{3})$,

$$\mathrm{cn}(3^{1/4}\omega x|m^*) > \lambda^* \tag{21}$$

Explicitly,

$$x < \beta\chi^{-1/3} \tag{22}$$

where β is a number whose approximate value is 1.1129.

When $\chi \to 0$ the dominant terms of (A1) reduce (15) to

$$R = M(1+1/\mathbf{c}) + O(\chi^2) \tag{23}$$

and (Abramowitz and Stegun, 1975)

$$\mathbf{c} = \mathrm{sech}\, x + O(\chi^2) \tag{24}$$

Exceptionally choosing $a=2/M$, (23) becomes just (3) if terms $O(\chi^2)$ be disregarded. This is certainly legitimate when

$$2M \ll r \ll 2M/\chi \tag{25}$$

It is sometimes useful to represent R by its series in ascending powers of x. Inserting the power series for $\mathbf{c}$ in (15) one finds that

$$\frac{R}{2M} = \frac{(1-4\lambda+\lambda^2)}{(1-\lambda)^2} + \frac{(1+\lambda)^2}{4(1-\lambda)^2}x^2 + \frac{(1+\lambda)^2(1+8\lambda+\lambda^2)}{48(1-\lambda)^2(1-4\lambda+\lambda^2)}x^4$$
$$+\frac{(1+\lambda)^2(1+52\lambda+138\lambda^2+52\lambda^3+\lambda^4)}{1440(1-\lambda)^2(1-4\lambda+\lambda^2)^2}x^5 + O(x^8) \tag{26}$$

This is consistent with (24), (25) since when $\lambda=0$ one has just the initial terms of the series for $\frac{1}{2}(1+\cosh x)$.

While $R(r,t)$ presents a reasonably simple appearance the same can hardly be said of $T(r,t)$. To begin with, if one uses (8) and (11) in (5) one immediately finds that

$$T_x = fR^2/\gamma \tag{27}$$

the sign on the right having been chosen arbitrarily. Inserting the explicit form of R as given by (15) there comes

$$T = -\frac{M^2k^3f}{2-k}\int^x \frac{(\mathbf{c}+1)^3\,dx}{(\mathbf{c}-\lambda)^2(\mathbf{c}-\kappa)}$$

The substitution $z := \mathbf{c}$ then leads to the result

$$T = -\frac{M^2k^3f}{h(2-k)}\int_{\mathbf{c}}^{1} \frac{(z+1)^3\,dz}{(z-\lambda)^2(z-\kappa)[(1-z^2)(m_1+mz^2)]^{1/2}} + \psi(t) \tag{28}$$

where $m_1 := 1-m$ and $\psi(t)$ is an arbitrary function of t. Now set $x=0$. Then $T=\psi(t)$ and $R_x=0$, while according to (18), (19) $R\neq 0$, $\gamma\neq 0$. Therefore $T_x\neq 0$ and (7) implies that $T_t=0$. Thus $\psi(t)$ is a constant which may be taken to be zero.

Evidently the integral in (28) is expressible in terms of known functions. It is, however, so cumbersome that it will suffice to set down its generic form. To this end, let the notation analogous to (16) be extended to the Jacobian elliptic functions sn and dn and to the incomplete elliptic integrals of the second and third kinds. For example, $\mathbf{\Phi}(n)$ stands for the incomplete elliptic integral of the third kind $\Phi(n;\xi|m)$ (cf. Abramowitz and Stegun, 1975). Then I find that

$$\begin{aligned} T/M^2f = {} & G_1\mathbf{\Pi}(g_1) + G_2\mathbf{\Pi}(g_2) + G_3[\operatorname{artanh}(g_3\mathbf{d}/\mathbf{c}) - \operatorname{artanh}(g_3)] \\ & + G_4[\operatorname{artanh}(g_4\mathbf{d}/\mathbf{c}) - \operatorname{artanh}(g_4)] + G_5\mathbf{sd}/(\mathbf{c}-\lambda) \\ & + G_6\mathbf{E} + G_7\xi \end{aligned} \tag{29}$$

where $G_1,\ldots,G_7$, $g_1,\ldots,g_4$ are all elementary algebraic functions of λ.

When x is sufficiently small one may also use (26) directly in (27). One so finds that

$$\begin{aligned} T = -\frac{2M(1-4\lambda+\lambda^2)^{3/2}}{(2\lambda)^{1/2}(1-\lambda)^2}\bigg(& x + \frac{(1+\lambda)^4}{24\lambda(1-4\lambda+\lambda^2)}x^3 \\ & + \frac{(1+\lambda)^4(3-4\lambda+34\lambda^2-4\lambda^3+3\lambda^4)}{192\lambda^2(1-4\lambda+\lambda^2)^2}x^5 + \cdots\bigg) \end{aligned} \tag{30}$$

The constancy of $a(t)$ can now be inferred as follows. Given $\lambda\,(0<\lambda<\lambda^*)$, choose a value of x so small that only dominant terms of (26) and (30) need be retained when finding the derivatives of R and T:

$$R =: \rho + \sigma x^2, \qquad T =: \tau x$$

Recall that here x stands for $\ln[a(t)r]$. Equation (7) now requires that

$$[(1-4\lambda+\lambda^2)^2\sigma\rho_\lambda - 2\lambda^2\tau\tau_\lambda]\lambda_t = 2\lambda^2\tau^2 a_t/a$$

The factor multiplying λ_t on the left vanishes identically, so that a_t must vanish.

3. $A(r, t)$ AND $B(r, t)$ IN EXPLICIT FORM

In view of (8) and (16) one has

$$A = Mkr^{-1}(\mathbf{c}+1)/(\mathbf{c}-\lambda) \tag{31a}$$

Next, according to (10)

$$B = A_t/fA$$

The explicit evaluation of A_t is an elementary but tedious task, for not only k and λ, but h and the parameter m of $\mathbf{c}$ are all functions of t. In particular one thus requires the derivative of $\mathbf{c}(\xi)$ with respect to m, with ξ held fixed:

$$2mm_1\partial\mathbf{c}(\xi)/\partial m = \mathbf{E} - m_1\xi - m\mathbf{sc}/\mathbf{d}$$

One finds eventually that

$$B = \frac{\lambda_t}{f}\left\{\frac{(\lambda+1)\mathbf{sd}}{(2-\lambda)(\mathbf{c}+1)(\mathbf{c}-\lambda)}\left[\frac{1-4\lambda+\lambda^2}{\lambda(1-2\lambda)}\left(\mathbf{E}-\frac{(1-\frac{1}{2}\lambda)\mathbf{sc}}{(1-\lambda^2)\mathbf{d}}\right)\right.\right.$$
$$\left.\left.+\frac{(7-4\lambda+\lambda^2)}{2(1-4\lambda+\lambda^2)}\xi\right]+\frac{1}{\mathbf{c}-\lambda}-\frac{(3-2\lambda+\lambda^2)}{(1-\lambda)(1-4\lambda+\lambda^2)}\right\} \tag{31b}$$

4. THE EVENT HORIZON

When one wishes to contemplate regions containing points of $\mathscr{H}$ one should use coordinates such that the metric is nonsingular on $\mathscr{H}$, for example, Kruskal-Szekeres coordinates (e.g., Buchdahl, 1981). The metrics (1) and (2), on the other hand, have coordinate singularities just on $\mathscr{H}$, i.e., when (for all values of t) $R = 2M$ and $r = \frac{1}{2}M$, respectively. What is the situation when the metric is (2a), with A and B given by (31)?

The angular variables θ, ϕ may be left aside, as usual. Then the points of $\mathscr{H}$ must evidently satisfy the condition

$$\mathbf{c} = \kappa \tag{32}$$

(32) can be put into a form which exhibits ξ explicitly as a function of λ

(i.e., of t):

$$\xi = F(\arccos \kappa | m) \tag{33}$$

where F denotes the elliptic integral of the first kind in the notation of Abramowitz and Stegun (1975). (See Appendix B for some additional detail.)

Now, inspection of (31a) shows that A^2 is well behaved in a neighborhood of $\mathcal{H}$ and it is everywhere positive. With regard to B^2 the situation is not so straightforward. One may use (26) in the relation $B = R_t/fR$ to infer that for sufficiently small values of λ

$$fB/\lambda_t = -2 + 14\lambda^2 + \tfrac{112}{3}\lambda^3 + O(\lambda^4) \tag{34}$$

where (33) has been used. (Recall that $\lambda \to 0$ implies that $\xi \to 0$.) On the other hand, as $\lambda \to \lambda^*$ both $c - \lambda$ and $1 - 4\lambda + \lambda^2$ go to zero. Therefore the right-hand member of (31b) is dominated by the term containing the explicit factor ξ. Taking $\lambda^* - \lambda$ to be sufficiently small, one infers that

$$fB/\lambda_t \sim j(\lambda^* - \lambda)^{-2} \tag{35}$$

where the constant $j > 0$. (I find that $j \approx 0.62486$.) Since $M \neq 0$ set $M = \frac{1}{2}$ as a matter of convenience. Then $\chi = \frac{1}{2}f$ and according to (A1), (A2)

$$\lambda \sim \begin{cases} \frac{1}{2}f^2 & (f \to 0) \\ \lambda^*(1 - 3^{-1/2}f^{-2/3}) & (f \to \infty) \end{cases}$$

so that by (34), (35)

$$B \sim \begin{cases} -2f_t & (f \to 0) \\ j'f^{4/3}f_t & (f \to \infty) \end{cases} \tag{36}$$

where j' is a positive constant. If $B =: H(f)f_t$ it follows from (36) that H vanishes somewhere in the range of f, say, at $f = f_1$. If $f_1 = f(t_1)$ and r_1 is the corresponding value of r calculated from (33), the 2-metric $d\sigma^2 := -A^2\,dr^2 + B^2\,dt^2$ is singular at the point (r_1, t_1), a conclusion not vitiated by the presence of the factor f_t in B. More generally there evidently is a line along which $d\sigma^2$ has a coordinate singularity.

APPENDIX A

q, λ, h are monotonically increasing and m, k are monotonically decreasing functions of $\chi(\geq 0)$. $m(\chi)$ only varies over a very small range, viz. $m(0) = 1$, $m(\infty) = \frac{1}{4}(2 + \sqrt{3}) \approx 0.933$. Numerical values of the various parameters for selected values of χ are shown in the table below.

χ	0.1	0.2	0.5	1.0	2.0	5.0	10
q	0.193	0.355	0.682	1.000	1.379	2.000	2.592
λ	0.017	0.050	0.122	0.172	0.206	0.234	0.247
m	0.992	0.977	0.953	0.942	0.937	0.934	0.933
h	1.036	1.116	1.369	1.682	2.100	2.839	3.574
k	0.947	0.843	0.599	0.414	0.274	0.153	0.098

As $\chi \to 0$

$$\begin{aligned} q &= 2\chi - 8\chi^3 + O(\chi^5) \\ \lambda &= 2\chi^2 - 32\chi^4 + O(\chi^6) \\ m &= 1 - \chi^2 + 20\chi^4 + O(\chi^6) \\ h &= 1 + 4\chi^2 - 44\chi^4 + O(\chi^6) \\ k &= 1 - 6\chi^2 + 88\chi^4 + O(\chi^6) \end{aligned} \tag{A1}$$

while as $\chi \to \infty$, if $\omega := (2\chi)^{1/3}$

$$\begin{aligned} q &= \omega - \tfrac{1}{3}\omega^{-1} + O(\omega^{-3}) \\ \lambda &= \lambda^*(1 - 3^{-1/2}\omega^{-2}) + O(\omega^{-4}) \qquad (\lambda^* 2 - \sqrt{3}) \\ m &= m^* + O(\omega^{-4}) \qquad (m^* = \tfrac{1}{4}(2+\sqrt{3}) \\ h &= 3^{1/4}\omega + O(\omega^{-1}) \\ k &= (\sqrt{3} - 1)\omega^{-2} + O(\omega^{-4}) \end{aligned} \tag{A2}$$

The approximation

$$\lambda = \lambda^*\left[1 - \left(\frac{\lambda^* + 4\chi^2}{\lambda^* + 10\chi^2 + 48\sqrt{3}\chi^4}\right)^{1/3}\right] \tag{A3}$$

which is based solely on (A1) and (A2) gives a simple heuristic picture of the dependence of λ on χ. It is numerically satisfactory.

APPENDIX B

According to (33), ξ is a single-valued function of λ, but x is not: as λ goes from λ to λ^*, ξ increases monotonically from 0 to $\xi^* \approx 1.845538$, but h^{-1} decreases from 1 to 0. ξ may be written as a series in ascending powers of $\lambda^{1/2}$:

$$\xi = (8\lambda)^{1/2}[1 + \tfrac{2}{3}\lambda + \tfrac{19}{30}\lambda^2 + O(\lambda^3)] \tag{B1}$$

This may be used to construct an elementary, rather satisfactory, approximation to ξ in closed form, i.e.,

$$\xi=(8\lambda)^{1/2}(1+\tfrac{2}{3}\lambda+\tfrac{19}{30}\lambda^2+\tfrac{151}{80}\lambda^3) \tag{B2}$$

the value of the coefficient of λ^3 being such as to ensure that when $\lambda=\lambda^*$ (B2) reproduces ξ^* with six-figure accuracy.

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